Let $G = (\{1, \ldots, n\}, E)$ be a simple graph on $n$ vertices.

**Definition.** A $G$-partial matrix is the image of a symmetric matrix under the projection

$$\pi_G : \text{Sym}_n(\mathbb{R}) \to \mathbb{R}^n \oplus \mathbb{R}^{|E|}$$

$$A = (a_{ij})_{\{i, j\} \in E}$$

**Example.**

$$a_{11} a_{12} a_{13} a_{14} a_{21} a_{22} a_{23} a_{24} a_{31} a_{32} a_{33} a_{34} a_{41} a_{42} a_{43} a_{44}$$

**Question.** Given a $G$-partial matrix, does there exist a positive semidefinite symmetric completion?
Symmetric matrix completion

\[
\begin{pmatrix}
1 & ? & -3 \\
? & 7 & ? \\
-3 & ? & -2
\end{pmatrix}
\]

**Constraints:**

1. Rank (smallest possible, fixed, …)
2. Positive semidefinite

\[
\begin{pmatrix}
* & ? & * \\
? & * & ? \\
* & ? & *
\end{pmatrix}
\]
**Spectral Theorem**

**Definition.** A real symmetric matrix is positive semidefinite if all of its real eigenvalues are nonnegative.

**Theorem.** A real symmetric matrix $M$ is positive semidefinite if and only if there exists a real matrix $B$ such that $M = B^T B$.

Real symmetric matrices $M$ are in one-to-one correspondence with real quadratic forms $Q_M = (x_1, x_2, \ldots, x_n)M(x_1, x_2, \ldots, x_n)^T$.

**Example.**

\[
M = \begin{pmatrix}
1 & 7 & -2 & 3 \\ 
7 & 1 & 5 & -13 \\ 
-2 & 5 & 1 & 11 \\ 
3 & -13 & 11 & 1 \\
\end{pmatrix}
\]

\[
Q_M = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 14x_1x_2 - 4x_1x_3 + 6x_1x_4 + 10x_2x_3 - 26x_2x_4 + 22x_3x_4
\]
Nonnegative Quadratic Forms and Sums of Squares

**Theorem.** A quadratic form $Q_M = x^T M x$ is positive semidefinite if and only if it is a sum of squares $Q_M = (Bx)^T (Bx)$.

**Example.**

$74x_1^2 + 122x_2^2 + 173x_3^2 + 144x_1x_2 + 202x_1x_3 + 282x_2x_3 = (5x_1 - x_2 + 2x_3)^2 + (7x_1 + 11x_2 - 13x_3)^2$

$$\begin{pmatrix} 74 & 72 & 101 \\ 72 & 122 & 141 \\ 101 & 141 & 173 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ -1 & 11 \\ 2 & -13 \end{pmatrix} \begin{pmatrix} 5 & -1 & 2 \\ 7 & 11 & -13 \end{pmatrix}$$

Smallest number of squares in $Q_M = \ell_1^2 + \ldots + \ell_r^2 = \text{rank of } M$
Theorem (Sylvester’s criterion). A symmetric matrix is positive semidefinite if and only if all of its principal minors are nonnegative.

Example.

\[ M = \begin{pmatrix} 1 & 2 & ? \\ 2 & 1 & ? \\ ? & ? & 7 \end{pmatrix} \quad Q_M = x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + ?x_1x_3 + ?x_2x_3 \]

\[ Q_M(1, -1, 0) = 1 + 1 - 4 = -2 \]

Example.

\[ \begin{pmatrix} 1 & 1 & ? & -1 \\ 1 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{pmatrix} \]
Nonnegative Quadratic Forms and Sums of Squares

Theorem. A quadratic form \( Q = \mathbf{x}^T \mathbf{M} \mathbf{x} \) is positive semidefinite if and only if it is a sum of squares \( Q = (\mathbf{Bx})^T \mathbf{Bx} \).

Example. \[ \begin{bmatrix} 1 & ? & ? \\ ? & 2 & 1 \\ ? & 1 & ? \end{bmatrix} \]

Smallest number of squares in \( Q = \ell_1 \ldots \ell_r \)

Obvious Necessary Condition

Theorem (Sylvester's criterion). A symmetric matrix is positive semidefinite if and only if all of its principal minors are nonnegative.

Example. \[ \begin{bmatrix} 1 & ? & ? \\ ? & 0 & 1 \\ ? & 1 & ? \end{bmatrix} \]

Aside: diagonal entries

Patterns and graphs
Complete principal minors: Clique

Let $G = (\{1, 2, 3, 4\}, E)$ be a simple graph on $n$ vertices.

**Example.** Let $\pi : \mathbb{R}^{nn} \to \mathbb{R}^{nn}$ be a symmetric matrix under the projection $\pi$. The partial matrix is the image of a symmetric matrix under the projection $\pi$.
Stanley-Reisner Ideals

**Definition.** The **Stanley-Reisner ideal** associated to $G = ([n], E)$ is the square-free monomial ideal

$$I_G = \langle x_i x_j : \{i, j\} \notin E \rangle \subset \mathbb{R}[x_1, x_2, \ldots, x_n] .$$

**Proposition.** (1) G-partial matrices $M$ are in one-to-one correspondence with residue classes of quadratic forms $Q_M$ in $\mathbb{R}[x_1, x_2, \ldots, x_n]/I_G = \mathbb{R}[X_G]$. 
(2) A G-partial matrix $M$ has a positive semidefinite completion if and only if the corresponding quadratic form $Q_M$ is a sum of squares of linear forms modulo $I_G$.

**Example.**

$$Q_M = x_1^2 + x_2^2 + x_3^2 + x_4^2 + 14x_1x_2 + 6x_1x_4 + 10x_2x_3 + 22x_3x_4 + ?x_1x_3 + ?x_2x_4$$

$$I_G = \langle x_1x_3, x_2x_4 \rangle$$

$$M = \begin{pmatrix} 1 & 7 & ? & 3 \\ 7 & 1 & 5 & ? \\ ? & 5 & 1 & 11 \\ 3 & ? & 11 & 1 \end{pmatrix}$$
Let $S = \mathbb{R}[x_1, x_2, \ldots, x_n]$.

**Definition.** A **minimal free resolution** of a graded $S$-module $M$ is an exact sequence

$$
\ldots \xrightarrow{\delta_{t+1}} F_t \xrightarrow{\delta_t} F_{t-1} \xrightarrow{\delta_{t-1}} \ldots \xrightarrow{\delta_1} F_0 \rightarrow M \rightarrow 0,
$$

where the $F_i$ are free $S$-modules and the image of the map $\delta_i$ is contained in the submodule $(x_1, x_2, \ldots, x_n)F_{i-1}$ of $F_{i-1}$ for all $i \geq 1$.

**Example.** Consider $M_1 = \langle x_1x_4, x_2x_4 \rangle$ and $M_2 = \langle x_1x_3, x_2x_4 \rangle$.

$$
0 \rightarrow S \xrightarrow{(x_2,-x_1)^T} S^2 \xrightarrow{(x_1x_4,x_2x_4)} M_1 \rightarrow 0
$$

$$
0 \rightarrow S \xrightarrow{(x_2x_4,-x_1x_3)^T} S^2 \xrightarrow{(x_1x_3,x_2x_4)} M_2 \rightarrow 0
$$

are minimal free resolutions of $M_1$ and $M_2$.

**Definition.** A homogeneous ideal $I \subset S$ is **2-regular** (Castelnuovo-Mumford) if it is generated by quadrics and the entries of $\delta_i$ have degree 1 for all $i \geq 1$.

**Theorem (Fröberg).** The ideal $I_G$ is 2-regular if and only if $G$ is chordal.
The Hankel Spectrahedron

**Definition.** The convex dual cone of $\Sigma_G$ is the **Hankel spectrahedron** of $G$.

**Proposition.**

$$\Sigma_G^\vee = S_+ \cap (\ker(\pi_G))^\perp$$

**Proof.**

$$\Sigma_G^\vee = (\pi_G(S_+))^\vee = S_+ \cap (\ker(\pi_G))^\perp$$

**Question:** What are the extreme rays of $\Sigma_G^\vee$?

**Proposition.** Extreme rays of $\Sigma_G^\vee$ that have rank 1 are in one-to-one correspondence with points $x \in \mathcal{V}(I_G) = X_G$.

**Proof.** $R = xx^T \in \Sigma_G^\vee$, $M \in \ker(\pi_G)) \cong I_G$:

$$0 = \langle M, R \rangle = \text{tr}(MR) = \text{tr}(Mxx^T) = x^TMx = Q_M(x)$$
Proposition. Let $G = ([m], E)$ be the $m$-cycle. The matrix
\[
\begin{pmatrix}
\frac{m-2}{m-1} & -1 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{m-1} \\
-1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & \ldots & \vdots & \vdots & \vdots \\
\vdots & 0 & -1 & 2 & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\
\frac{1}{m-1} & 0 & 0 & 0 & \ldots & 0 & -1 & \frac{m-2}{m-1}
\end{pmatrix}
\]

is an extreme ray of $\Sigma^\vee_G$. It has rank $m - 2$. It certifies that
\[
\begin{pmatrix}
1 & 1 & ? & \ldots & ? & \ldots & ? & -1 \\
1 & 1 & 1 & ? & \ldots & ? & ? & ? \\
? & 1 & 1 & 1 & \ldots & ? & ? & ? \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
? & ? & ? & ? & \ldots & 1 & 1 \\
-1 & ? & ? & ? & \ldots & 1 & 1
\end{pmatrix}
\]
does not have a positive semidefinite completion even though every complete principal minor is positive semidefinite.
Hankel index and bpf linear series

**Definition.** The **Hankel index** of $G$ is the smallest $r > 1$ such that $\Sigma_G^\vee$ has an extreme ray of rank $r$.

Let $R \in \Sigma_G^\vee = S_+ \cap (\ker(\pi_G))^\perp$ be an extreme ray.

1. The kernel of $R$ is a linear series on $X_G$: $\ker(R) \subset \mathbb{R}[x_1, x_2, \ldots, x_n]_1$.
2. If $\text{rk}(R) > 1$, then $\ker(R)$ is a base-point free linear series on $X_G$. (Blekherman)
3. $\langle \ker(R) \rangle_2$ is contained in $R^\perp$.

**Theorem (Blekherman, Sinn, Velasco).** The Hankel index of $G$ is at least the (Green-Lazarsfeld index of $I_G$) + 1.

**Theorem (Eisenbud, Green, Hulek, Popescu).** The Green-Lazarsfeld index of $I_G$ is the (smallest length of a chordless cycle in $G$) - 2.

**Theorem.** Let $m$ the smallest length of a chordless cycle in $G$. A $G$-partial matrix $M$ has a positive definite completion if and only if

1. every complete principal minor of $M$ is positive, and
2. $M$ has a positive semidefinite completion of rank at least $n - m + 2$. 

References
MLT vs. GCR

Let $G = ([n], E)$ be a labeled simple graph.

**Definition.** (1) The maximum likelihood threshold of $G$ is the smallest whole number $k$ such that there exists a positive definite matrix $D$ with $\pi_G(M) = \pi_G(D)$ for almost all positive semidefinite matrices $M$ of rank at most $k$.

(2) The generic completion rank of $G$ is the smallest whole number $k$ such that $\dim(\pi_G(V_k)) = n + \#E$, where $V_k$ is the variety of matrices of rank at most $k$.

**Proposition (Sard’s Theorem and Implicit Function Theorem).** $\text{gcr}(G) \geq \text{mlt}(G)$

**Theorem (Blekherman, Sinn).**

\[
\text{gcr}(K_{m,m}) = m
\]
\[
\text{mlt}(K_{m,m}) = \min \left\{ k: \binom{k+1}{2} \geq 2m \right\} \in O(2\sqrt{m}).
\]
Gram Dimension (SOS length)

**Definition (Laurent, Varvitsiotis).** The **Gram dimension** of a labeled simple graph \( G = ([n], E) \) is the smallest whole number \( k \) such that for any positive semidefinite matrix \( M \in S_+ \), there exists another \( M' \in S_+ \) of rank at most \( k \) such that \( \pi_G(M) = \pi_G(M') \).

**Theorem (Laurent, Varvitsiotis).** (1) \( k \leq 3: \) The Gram dimension of \( G \) is at most \( k \) if and only if \( G \) has no \( K_{k+1} \) minor.
(2) The Gram dimension of \( G \) is at most 4 if and only if \( G \) has no \( K_5 \) and \( K_{2,2,2} \) minors.

**Example.** The Gram dimension of any \( m \)-cycle is 3.